MATH 292 – Test 2 SU26

Name: Keitaro Miyai

$\frac{1}{0} \to \infty$

1.

Find the limit, if it exists. (If the limit does not exist, state DNE and justify your conclusion.)

$$\lim_{(x,y,z)\to(0,0,0)} \frac{2xy - yz + xz}{x^2 + y^2 + z^2}$$

Along $x = y = z$,

$$\lim_{(x,y,z)\to(0,0,0)} \frac{2xy - yz + xz}{x^2 + y^2 + z^2}$$

$$= \lim_{x\to 0} \frac{2x^2 - x^2 + x^2}{x^2 + x^2 + x^2}$$

$$= \lim_{x\to 0} \frac{2x^2}{3x^2}$$

$$= \frac{2}{3} \quad \cdots ①$$

Along $y = 0$ and $z = 0$,

$$\lim_{x\to 0} \frac{0 - 0 + 0}{x^2 + 0 + 0}$$

$$= 0 \quad \cdots ②$$

Since ① and ②,

the limits along different paths are different,

the limit DNE. ☒

2.

Use implicit differentiation to find

$$\frac{\partial z}{\partial x} \text{ and } \frac{\partial z}{\partial y}.$$

$$e^{yz} + x^2 z = 4xy^3 + z^2$$

$e^{yz} + x^2 z = 4xy^3 + z^2$

$e^{yz} + x^2 z - z^2 = 4xy^3$

$\frac{\partial}{\partial x}(e^{yz} + x^2 z - z^2) = \frac{\partial}{\partial x}(4xy^3)$

$ye^{yz} \cdot \frac{\partial z}{\partial x} + 2xz + x^2 \frac{\partial z}{\partial x} - 2z\frac{\partial z}{\partial x} = 4y^3$

$\frac{\partial z}{\partial x}(ye^{yz} + x^2 - 2z) = 4y^3$

$\therefore \frac{\partial z}{\partial x} = \frac{4y^3}{ye^{yz} + x^2 - 2z}$

$e^{yz} + x^2 z - z^2 = 4xy^3$

$\frac{\partial}{\partial y}(e^{yz} + x^2 z - z^2) = \frac{\partial}{\partial y}(4xy^3)$

$(z + y\frac{\partial z}{\partial y})e^{yz} + x^2\frac{\partial z}{\partial y} - 2z\frac{\partial z}{\partial y} = 12xy^2$

$\frac{\partial z}{\partial y}(ye^{yz} + x^2 - 2z) = 12xy^2 - ze^{yz}$

$\frac{\partial z}{\partial y} = \frac{12xy^2 - ze^{yz}}{ye^{yz} + x^2 - 2z}$

3.

Find an equation of the tangent plane to the surface

$$z = x^2 y$$

at the point

$$(2, -1, -4).$$

$f_x = \frac{\partial z}{\partial x} = 2xy$

$f_y = \frac{\partial z}{\partial y} = x^2$

$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

$z - z_0 = 2x_0 y_0 (x - x_0) + x_0^2 (y - y_0)$

when $(x_0, y_0, z_0) = (2, -1, -4)$

$z + 4 = 2 \cdot 2(-1)(x - 2) + 2^2 \cdot (y + 1)$

$z + 4 = -4x + 8 + 4y + 4$

$4x - 4y + z = 8$

4.

Find the local maximum value(s), local minimum value(s), and saddle point(s) of

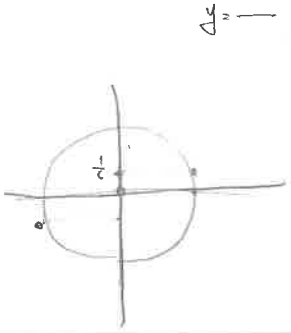
$$f(x,y) = x^3 - 3x + 3xy^2.$$

$f_x = 3x^2 - 3 + 3y^2$ $\quad f_y = 6xy$

$f_{xx} = 6x$ $\quad f_{yy} = 6x$

$f_{xy} = 6y$

$$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases}$$



$$\Leftrightarrow \begin{cases} 3x^2 - 3 + 3y^2 = 0 \\ 6xy = 0 \end{cases}$$

$\Leftrightarrow (x,y) = (0,1)$
$(0,-1)$
$(1,0)$
$(-1,0)$

① When $(x,y) = (0, \pm 1)$,

$D = f_{xx} \cdot f_{yy} - (f_{xy})^2$

$= 36x^2 - 36y^2$

$= -36 < 0 \Rightarrow$ Saddle point

$f(0, \pm 1) = 0$

② When $(x,y) = (1,0)$,

$D = 36x^2 - 36y^2$

$= 36 > 0$,

$f_{xx} = 6x = 6 > 0$.

So the point is local minimum.

$f(1,0) = -2$

② When $(x,y) = (-1,0)$,

$D = 36x^2 - 36y^2$

$= 36 > 0$,

$f_{xx} = 6x = -6 < 0$

So the point is local maximum.

$f(-1,0) = 2$

Since ① and ② and ③,

Local maximum value: $f = 2$, $(x,y) = (-1,0)$

Local minimum value: $f = -2$, $(x,y) = (1,0)$

Saddle point: $f = 0$, $(x,y) = (0, \pm 1)$

5.

Find the maximum rate of change of the function at the indicated point and the direction in which this maximum rate of change occurs.

$$f(x,y) = \ln(x^2 + y^2)$$

at the point

$$(1, \sqrt{3}).$$

$\nabla f = \left\langle \frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2} \right\rangle$

Maximum rate of change: $\|\nabla f\| = \left\| \left\langle \frac{2}{4}, \frac{2\sqrt{3}}{4} \right\rangle \right\|$

$= \left\| \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle \right\|$

$= \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$

$= 1$ ∴ Maximum rate of change: 1

The direction: $\frac{\nabla f}{\|\nabla f\|} = \frac{\left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle}{1}$

$= \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$ (it means $\frac{\pi}{3}$ above the X-axis.)

6.

Use the chain rule to find

$$\frac{dz}{dt'}$$

where

$$z = \frac{x-y}{2x+y}, x = e^{2t}, y = e^{-t}.$$

$$\frac{\partial z}{\partial x} = \frac{1\cdot(2x+y) - (x-y)2}{(2x+y)^2}$$

$$= \frac{2x+y-2x+2y}{(2x+y)^2}$$

$$= \frac{3y}{(2x+y)^2}$$

$$\frac{\partial z}{\partial y} = \frac{-1\cdot(2x+y) - (x-y)\cdot 1}{(2x+y)^2}$$

$$= \frac{-3x}{(2x+y)^2},$$

$$\frac{dx}{dt} = 2e^{2t}, \quad \frac{dy}{dt} = -e^{-t}.$$

Since then, $\langle 2x \quad > < -x \quad y >$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}.$$

$$= \frac{3y}{(2x+y)^2}\cdot 2e^{2t} + \frac{-3x}{(2x+y)^2}\cdot(-e^{-t})$$

$$= \frac{3xe^{-t} + 6ye^{2t}}{(2x+y)^2}$$

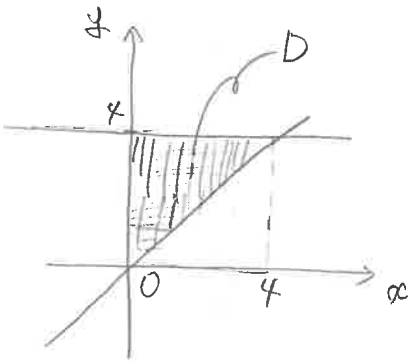
$$= \frac{3e^{2t}\cdot e^{-t} + 6e^{-t}\cdot e^{2t}}{(2e^{2t} + e^{-t})^2}$$

$$= \frac{3e^{t} + 6e^{t}}{(2e^{2t} + e^{-t})^2} = \frac{9e^{t}}{(2e^{2t}+e^{-t})^2}$$ //

7.

Set up iterated integrals for **both** orders of integration ($dy\,dx$ and $dx\,dy$). **DO NOT EVALUATE!** Draw a sketch of the region as part of your work.

$$I = \iint_D y\,e^{xy}\,dA$$



where D is the region bounded by

$$y = x, y = 4, x = 0.$$

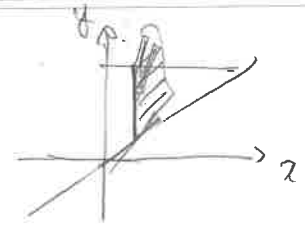
$D = \{(x,y) \mid 0 \le x \le 4,\ x \le y \le 4\}$

or

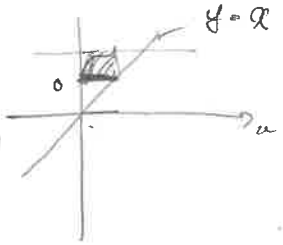
$D = \{(x,y) \mid 0 \le x \le y,\ 0 \le y \le 4\}$

Since then,

$$I = \int_0^4 \int_x^4 y\,e^{xy}\,dy\,dx,$$



$$I = \int_0^4 \int_0^y y\,e^{xy}\,dx\,dy.$$



8.

Find the volume of the solid enclosed by the paraboloid

$$z = 1 + x^2 + (y-1)^2 \quad > 0$$

and the planes

$$z = 0, x = -1, x = 1, y = -1, y = 3.$$

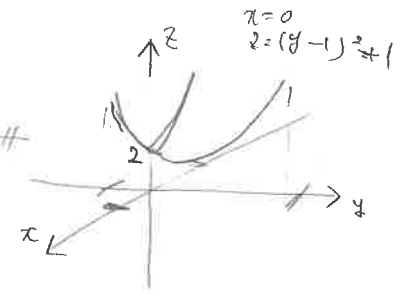
$-1 < x < 1, \quad -1 < y < 3, \quad 0 < z < 1 + x^2 + (y-1)^2$

Set up (but do not evaluate) the double integral needed to compute the volume.

$$V = \int_{-1}^{3}\int_{-1}^{1} z \, dx \, dy \qquad x^2+y$$

$$= \int_{-1}^{3}\int_{-1}^{1} \{1 + x^2 + (y-1)^2\} \, dx \, dy$$

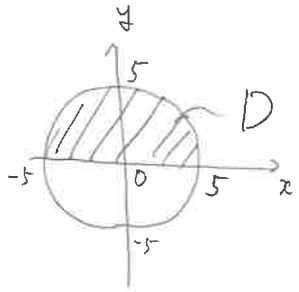
$$= \int_{-1}^{1}\int_{-1}^{3} \{1 + x^2 + (y-1)^2\} \, dy \, dx.$$



9.

Evaluate the integral by changing to polar coordinates.

$$I = \iint_D 3x^2\,dA$$



where D is the upper semicircle centered at the origin with radius 5.

$x = r\cos\theta$, $0 \le r \le 5$, $0 \le \theta \le \pi$,

$dA = r\,dr\,d\theta$,

Sine them,

$$I = \int_0^{\pi}\int_0^{5} 3(r\cos\theta)^2\, r\,dr\,d\theta.$$

$$= 3\int_0^{\pi}\int_0^{5} r^3\cos^2\theta\,dr\,d\theta$$

$$= 3\int_0^{\pi}\left[\frac{1}{4}r^4\cdot\cos^2\theta\right]_{r=0}^{r=5} d\theta$$

$$= \frac{3}{4}\int_0^{\pi} 625\cos^2\theta\,d\theta$$

$$= \frac{1875}{4}\int_0^{\pi}\left(\frac{\cos 2\theta}{2} + \frac{1}{2}\right) d\theta$$

$$= \frac{1875}{4}\left[\frac{1}{4}\sin 2\theta + \frac{1}{2}\theta\right]_0^{\pi}$$

$$= \frac{1875}{4}\left(\frac{1}{2}\pi\right) = \frac{1875}{8}\pi$$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$= 2\cos^2\theta - 1$

$\frac{\cos 2\theta + 1}{2} = \cos^2\theta$

$2\pi = 4y^2$

$$3 \times \left[\frac{1}{4}r^4\right]_0^5 \times \left[\frac{1}{4}\sin 2\theta + \frac{1}{2}\theta\right]_0^{\pi}$$

$$= 3 \times \frac{1}{4} \times 625 \times \frac{1}{2}\pi$$

$$= \frac{1875}{8}\pi$$