NO. Math-194-1

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# 用語集

System of linear equations　連立一次方程式

Coefficient　係数

Constant　定数

Matrix　行列

Elementary row operations　行基本変形

Row →　Column ↓　行，列

Multipy かける　devide 割る

coefficent matrix　係数行列

augmented matrix　拡大係数行列

RE = Row-Echelon Form　階段行列

$$\left|\begin{array}{cccc} 1 & 2 & 1 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right|$$

RRE = Reduced Row-Echelon Form　行簡約行列

$$\left|\begin{array}{cccc} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right|$$

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1.2

A system of linear eqs. where all the constants are zero is called homogeneous.

$a_{11}x_1 + a_{12}x_2 \cdots + a_{1n} = 0$

$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn} = 0$ この方程式は $x_1 x_2 \cdots x_n = 0$ を解に持つので、必ず解ける。

1.3 Applications

座標平面上でn個の点が与えられたとき、n-1次式でそれら全ての点を通る式ができる。

Given n points in the plane, we can find a n-1 degree or less polynominal that passes thru the n points with a system of eqs.

<帰納法でも行列式でも解けるよ>

2.1 Matrices

We can write matrices in 3 ways

1) $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = A$ 2) A, B, C 3) $[a_{ij}]$, $[b_{ij}]$

If $A = [a_{ij}]$, $B = [b_{ij}]$ s.t. $A + B = [a_{ij} + b_{ij}]$ and A, B are same size.

(積) Scalar Multiplication $cA = c[a_{ij}] = [cA_{ij}]$

Matrix Multiplication

Let $A = [a_{ij}]$, $B = [b_{ij}]$ where A is a $m \times n$ matrix and B is a $n \times p$ matrix

then the product AB is a $m \times p$ matrix

where $AB = [c_{ij}]$ where $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

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$a_1\ a_2\ a_3\ a_4 \cdots a_n$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \\ \vdots & & & \\ a_{m1} & & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = b$$ のとき、

線形結統

$b = x_1a_1 + x_2a_2 + \cdots + x_na_n$ を the linear combination という。

## 2.2 行列の計算

1) 交換法則 Commutative Law of Addition
2) 結合法則 Associative Law of Addition
3) $c(dA) = cdA$ Associative Law for Multiplication
4) $1 \cdot A = A$ Existance of Multiplication
5) $c(A+B) = cA + cB$ Distributive Law
6) $(c+d)A = cA + dA$ Distributive Law

$A^T$ は A の転置行列 Transpose
$A^{-1}$ は A の逆行列 Inverse
$AA^{-1} = A^{-1}A = I$ になる。

1) $(A^T)^T = A$
2) $(A+B)^T = A^T + B^T$
3) $(cA)^T = c(A^T)$
4) $(AB)^T = B^TA^T$ $(AB)^T \neq A^TB^T$

## 2.3 Inverse of Matrix

$ax - b$ のとき、$x = a^{-1}b$ となる。

Theorem If a matrix is invertive, then its inverse is unique.

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逆行列

$A=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$ and $B=\begin{bmatrix}0 & -1\\ 0 & 1\end{bmatrix}$ are inverses. $AB=I=BA$

行列同士をかけたら $I$ になるとき、それぞれもう互いの逆行列という。
$AB=I$ になるとき $A,B$ が正方行列なら、$BA=I$ も同時に成立する。

If $a_{11}a_{22}-a_{21}a_{12}=0$ then $A^{-1}$ does not exist

If $a_{11}a_{22}-a_{21}a_{12}\neq 0$, then $A^{-1}$ exists and

- $A^{-1}=\dfrac{1}{a_{11}a_{22}-a_{12}a_{21}}\begin{bmatrix}a_{22} & -a_{12}\\ -a_{21} & a_{11}\end{bmatrix}$

2.4 A $n\times n$ matrix is an elementary matrix if it can be obtained from $I_n$ by 1 elementary row operations.

基本行列 $E$ と行列 $A$ の積 $EA$ を計算すると、$E$ が生み出された操作を $A$ にほどこすことになる。

Def Let $A, B$ be $m\times n$ matrices. Matrix $B$ is called Row-equivalent to $A$ matrix if there exists a finite number of elementary matrices $E_1, E_2, E_3 \cdots E_k$ such that $B=E_k\cdots E_2E_1A$ (求同値)

{ 右から正則行列をかける… 行基本変形
{ 左から正則行列をかける… 列基本変形

Theorem If $A$ is an $m\times n$ matrix, then the following are equivalent.
1) $A$ is invertible.
2) $A\vec{x}=\vec{b}$ has a unique solution for every $\vec{b}$.
3) $A\vec{x}=\vec{0}$ has only a trival solution
4) $A$ is row-equivalent to $I_n$.
5) $A$ can be written as a product of elementary matrices.

# 4. Vector Spaces ベクトル空間

ベクトル空間になるためのルール・公理 (Axioms)

1. Closure under addition. 足し算について閉じていること.
2. Closure under Scalar multiplication スカラー倍について閉じていること.
3. Commutative Property 交換法則 $u+v=v+u$
4. Addictive Identity ゼロベクトルが存在 $u+0=u$
5. Addictive Inverse 逆元が存在 $u+(-u)=0$
6. Distributive Property 1. 分配法則1 $c(u+v)=cu+cv$
7. Distributive Property 2 分配法則2 $(c+d)u=cu+du$
8. Associative Property of Scalar Multiplication $c(du)=(cd)u$
9. Scalar Identity $1\cdot u=u$

## 4-3. Subspace 部分空間.

1. ゼロベクトルを含むか $0\in W$
2. 足し算に閉じているか. $u, v\in W \Rightarrow u+v\in W$
3. スカラー倍で閉じているか $u\in W \Rightarrow cu\in W$

$\mathbb{R}^n$ はベクトル空間であることが証明されているため. $\mathbb{R}^n$ 内の特定の部分空間を扱う際、上記3つの条件のみ調べれば良い。

$\mathbb{R}^n \Rightarrow$ 実数 n コのベクトル. 例: $\mathbb{R}^3=\langle a, b, c\rangle$

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$W=\{0\}$ ← 部分空間（最小） $W=V$ ← 部分空間（最大）

Any other subspaces of $V$ are called proper subspaces of $V$.

⇒ベクトル空間 $V$ のうち、$V$ そのものではない部分空間。＜真部分空間＞

All subspaces of $\mathbb{R}^2$

1) $\{\langle 0,0\rangle\}$
2) Set of all lines passing through the origine.
3) $\mathbb{R}^2$

## 4-4 Spaning Sets and Linear Independance

Def A vector $v \in V$, is called a linear combination of vectors $v_1, v_2, \cdots v_n \in V$ if $v = c_1v_1 + c_2v_2 + \cdots + c_nv_n$,
Also, we say $v$ is linearly dependent on $v_1, v_2, \cdots v_n$

$v$ は $v_k$ の線形接続であり、線形従属である。

# Chapter 5 Inner Product Spaces

## Section 5.1

**Def** The length, norm or Magnitude of a vector $v \in \mathbb{R}^n$, $v = \langle v_1, v_2, \cdots v_n \rangle$ is given as $\|v\| = \sqrt{v_1^2 + v_2^2 \cdots + v_n^2}$

If $\|v\| = 1$, then $v$ is called a unit vector.
For all $v \in \mathbb{R}^n$, $\|v\| \geq 0$
Further, $\|v\| = 0$ iff $v = 0$

- Distance Between two vectors.
  $u \in \mathbb{R}^n$, $v \in \mathbb{R}^n$
  $d = \sqrt{\|u - v\|}$ $d \geq 0$, $d = 0$ when $u = v$.

**Theorem** Propaties of Dot Product

Let $u, v, w \in \mathbb{R}^n$, $c$ a scalar, then the following is true.

1) $v \cdot u = u \cdot v$
2) $u \cdot (v + w) = u \cdot v + u \cdot w$
3) $c(u \cdot v) = cu \cdot v = u \cdot cv$
4) $v \cdot v = \|v\|^2$
5) $v \cdot v \geq 0$ and $v \cdot v = 0$ iff $v = 0$

$\mathbb{R}^n$ with vector addition, scalar multiplication and the dot product, is called Euclidean n-space.

**Theorem** Cauchy - Scharwz Inequality
Let $u, v \in \mathbb{R}^n$, then
$|u \cdot v| \leq \|u\|\|v\|$

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## 5.2 Inner Product Spaces

A vector space $V$, together with an inner product is called an inner product space.

1) $\langle u, v \rangle = \langle v, u \rangle$
2) $\langle u, v+w \rangle = \langle u, v \rangle + \langle u, w \rangle$
3) $c\langle u, v \rangle = \langle cu, v \rangle$
4) $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0$ iff $v = 0$

これらを満たす全てのベクトル空間が Inner product space.

$\langle u, v \rangle = \sum_{k=1}^{n} c_k u_k v_k$ の形で、$c_k \leq 0$ が1つでもあるなら Inner product space ではなくなる。

Inner Product Space では、$c_k > 0$ が全ての $k$ ($1 \leq k \leq n$) で成立する。

Theorem Properties of Inner Products

: Inner Product Space

Let $u, v$ be vectors from an I.P.S., and $c \in \mathbb{R}$,

1) $\langle 0, v \rangle = \langle v, 0 \rangle = 0$
2) $\langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$
3) $\langle u, cv \rangle = c\langle u, v \rangle$

**Def** Let $u, v \in$ I.P.S., then

1) $\|u\| = (\langle u, u \rangle)^{1/2}$
2) The distance between $u, v$ is given by
   $d(u, v) = \|u - v\|$
3) If $u, v \neq 0$, then the angle between them $\theta$ is
   $\cos\theta = \dfrac{\langle u, v \rangle}{\|u\|\|v\|}$, $0 \leq \theta \leq \pi$
4) $u, v$ are orthogonal if $\langle u, v \rangle = 0$

If $\|u\| = 1$, then $u$ is called unit vector.
Given any $v \in$ I.P.S., $v \neq 0$ then $u = v/\|v\|$ is called the unit vector in the direction of $v$.

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## 6.1 Intro to Linear Transformations

A map is a function from one set to another set.

We will consider maps from a vector space to a vectorspace:

Notation: $T: V \to W$ T is a transformation (map) from $V$ to $W$.
V is called the domain
W is called the co-domain
If $v \in V$, $w \in W$, then $T(v) = w$ image of $v$ under $w$.

The set of all images of vectors in $v$ is called the range of T.

The